CS-151 Quantum Computer Science: Problem Set 3

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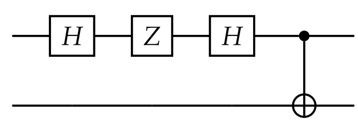
Guidelines: ***The deadline to return this problem set is 11.59pm on Wednesday, February 14th***. *Remember that you can collaborate with each other in the preliminary stages of your progress, but each of you must write their solutions independently. Submission of the problem set should be via Gradescope only. Best wishes!*

Problem 1 (15 points). *Recall SWAP is the operator that takes two labels $|ab\rangle$ for $a, b \in \{0, 1\}$ and swaps the order of these labels, i.e. outputs $|ba\rangle$. Let $A = c \cdot I + d \cdot SWAP$ where I and SWAP are given by their 4×4 matrix representation and $c, d \in \mathbb{C}$. Under what conditions for c and d, is the matrix A unitary?*

Problem 2 (20 points).

a. *Consider the following quantum circuit C:*



Calculate the matrix of the unitary operation U corresponding to C.

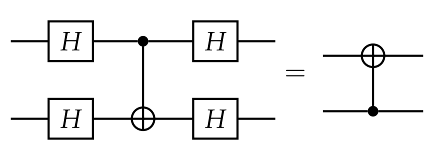
b. *Draw the inverse quantum circuit corresponding to U^{-1}.*

c. *A controlled-Z gate take 2 qubits as inputs, and gives 2 qubits as outputs. If the control qubit is $|0\rangle$, the gate does nothing to the target qubit. If the control qubit is $|1\rangle$, then Z is applied to the target. Show that a controlled-Z gate can be implemented using only CNOT and H gates. Draw the resulting circuit.*

(***Remark.*** *The H matrix transforms between the $\{\ |0\rangle, |1\rangle\ \}$ (computational) basis and the $\{\ |+\rangle, |-\rangle\ \}$ basis.)*

Problem 3 (15 points).

a) *In class, we saw how the CNOT gate acts on the computational basis. We also saw that H transforms between the $\{\ |0\rangle, |1\rangle\ \}$ and $\{\ |+\rangle, |-\rangle\ \}$ bases. Show that the following quantum circuits are equivalent.*



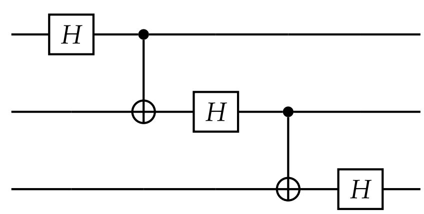
i.e., Show that given the same input states, they produce the same outputs.

b) *Verify that the action of CNOT on the* $\{\ |+\rangle, |-\rangle\ \}$ *basis is given by,*

$$
\begin{aligned}
|+\rangle\,|+\rangle &\mapsto |+\rangle\,|+\rangle \\
|-\rangle\,|+\rangle &\mapsto |-\rangle\,|+\rangle \\
|+\rangle\,|-\rangle &\mapsto |-\rangle\,|-\rangle \\
|-\rangle\,|-\rangle &\mapsto |+\rangle\,|-\rangle
\end{aligned}
$$

That is, it looks like the roles of target and control have been swapped.

c) *Consider the following quantum circuit,*



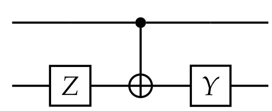
Calculate the output state if the input is given by the all-zeros state $|0\rangle\,|0\rangle\,|0\rangle$.

Problem 4. *Let* $|\psi\rangle = \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$ *and* $|\phi\rangle = \frac{1}{\sqrt{5}}|0\rangle - \frac{2}{\sqrt{5}}|1\rangle$.

a) *What is the probability that upon measuring* $|\psi\rangle \otimes |\phi\rangle$ *we obtain the same bits (i.e. we obtain* 00 *or* 11*).*

b) *Answer the same question as part (a) for* $(H \otimes H)(|\psi\rangle \otimes |\phi\rangle)$.

c) *First show that CNOT is the operator* $|0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes X$. *Use this expression to come up with an expression for the circuit below in terms of sums of tensor products (Hint: compute* $(I \otimes Z)CNOT(I \otimes Y)$*).*



d) *Let C be the circuit in part (c). Repeat part (a) for* $C(|\psi\rangle \otimes |\phi\rangle)$

Problem 5 (20 points)**.** *We introduced the tensor product operation* $\otimes$ *in a previous lecture. It is a useful mathematical tool that we will use often in this class. Below, you will prove that the tensor product preserves certain properties of the underlying matrices.*

a) *Show that the tensor product of two unitary operators is unitary.*

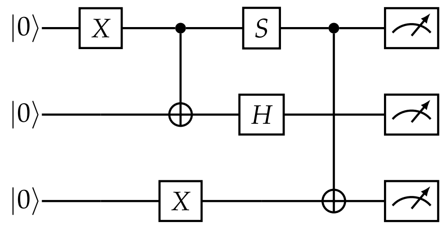
b) *Show that the tensor product of two Hermitian operators is Hermitian.*

c) *Show that the tensor product of two projectors is a projector.*

d) *show that the tensor product of two positive operators is positive.* ***[Extra credit]***

Problem 6 (15 points).

a) *Consider the following circuit. What is the final state? What is the probability of measuring* $|0\rangle$ *on each register?*



Problem 7 (Extra credit). *In class we discussed rotation around Z and Y axis. In this problem, we want to obtain a fundamental understanding of these operations. For a matrix A the exponential of the matrix* e^A *is defined by its Taylor series:* $e^A = I + A + A^2/2 + A^3/3! + \ldots$.

a) *Show that* $e^{-i\frac{\theta}{2}Y}$ *is unitary. (Hint: Find an expression for* $(e^A)^\dagger$. $e^{A+B} = e^A e^B$ *if* $AB = BA$.)

b) *Using the Taylor series show that* $e^{-i\frac{\theta}{2}Y} = \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix} = R_Y(\theta)$.

c) *Show that* $R_Y(\theta) = SR_X(\theta)S^{-1}$.

d) *Explain why you get the same result in parts (b) and (c).*